Phys 410
Fall 2015
Lecture #26 Summary
1 December, 2015

We continued the discussion of nonlinear mechanics, focusing on the periodically driven damped pendulum. The driven damped pendulum has an equation of motion for the generalized coordinate φ given by $\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2 \sin\varphi = \gamma\omega_0^2 \cos\omega t$, where β is the familiar damping frequency, $\omega_0^2 = g/\ell$ is the natural oscillation frequency of the pendulum, and $\gamma = F_0/mg$ is the dimensionless ratio of the sinusoidal forcing amplitude to the weight of the bob.

As the driving strength (parameterized by γ) is increased, there are solutions at sub-harmonics of the drive frequency. In other words, we have to wait for multiples of the drive period before the motion of the pendulum repeated itself. Note that this is different from harmonic generation, such as the third harmonic response discussed in the last lecture. As γ is increased there is a series of period doubling bifurcations, at ever decreasing intervals of γ. This sequence of bifurcations is described by the Feigenbaum number. This leads to the prediction of chaos (motion with period infinity) at a finite driving strength. This transition is observed in the solution for $\varphi(t)$.

An excellent way to distinguish periodic and chaotic systems is in terms of their sensitivity to initial conditions. Periodic systems (including those at subharmonics) have an insensitivity to initial conditions in the long-time behavior is dictated by the driving force alone, and identical for all initial conditions. Chaotic systems on the other hand have extreme sensitivity to initial conditions. Even very small differences in their initial state of motion become exponentially larger as time evolves, and eventually it is impossible to predict the state of one solution relative to another started with different initial conditions. These observations are summarized by the Lyapunov exponent (λ) for the growth (or decay) of the difference between two solutions ($\Delta\varphi(t)$) as $\Delta\varphi(t) \sim Ke^{\lambda t}$, where $K > 0$. Periodic systems have $\lambda < 0$, while chaotic systems have $\lambda > 0$.

We then discussed several methods to summarize the motion in terms of diagrams and plots. The first is a bifurcation diagram, which shows a stroboscopic sampling of the solutions $\varphi(t)$ and reveals periodic and chaotic motion pictorially. The next is the state space plot where the solution is plotted in the $\varphi(t)$, $\dot{\varphi}(t)$ plane with time as the parameter. This also reveals the periodic and chaotic motions quite distinctly. Finally we considered the Poincare section which is a stroboscopic sampling of the state space trajectory.

The bifurcation diagram summarizes the types of motion that are observed under different normalized driving strength γ. The long-time behavior of the pendulum angle $\phi(t)$ is sampled stroboscopically and plotted on the diagram. The stroboscope period corresponds to the drive period. The regions of period 1, period 2, period 4, and chaotic motion are clearly visible. One surprising result is that periodic motion can be seen in narrow windows of γ in the middle of chaotic solutions. Another surprise is that periodic motion can re-appear at larger driving strength, along with other bouts of period doubling and chaos.

We examined the solutions represented as trajectories in a two-dimensional state space described by $(\phi, \dot{\phi})$ as a parametric function of time. These trajectories go on to limit cycle curves for the periodic solutions, and form space-filling curves for chaotic solutions. The running trajectories in which the pendulum winds continuously are hard to represent on bifurcation diagrams and state-space plots. In this case it becomes useful to make Poincare sections that consist of stroboscopically sampled points from state space. These create fractal structures for chaotic solutions.

Finally we discussed some physical realizations of period doubling in diode circuits, neural activity and heat flow through a fluid. There is a direct analogy between the driven damped pendulum and the driven Josephson junction. Certain crystals of high temperature superconductors have an intrinsic Josephson effect between superconducting layers, and such systems act as collections of coupled driven damped pendula, and display an amazing variety of physical phenomena.

We then turned to a discussion of Special Relativity. We began by reviewing the Galilean transformation between inertial reference frames, and showed that Newton's second law of motion holds in the same form in all inertial reference frames. This result relies on the Galilean velocity addition formula between reference frames. However, it was discovered that Galilean invariance does not apply to Maxwell's equations (which are actually Lorentz invariant) by examining the measurement of the speed of light in a moving reference frame. The Michelson-Morley experiment showed that the measured speed of light is the same in all directions for all inertial observers. Hence there must be something more going on than simple Galilean transformations between reference frames.

Einstein made two postulates:

1) If S is an inertial reference frame and if a second frame S' moves with constant velocity relative to S, then S' is also an inertial reference frame.
2) The speed of light (in vacuum) has the same value *c* in every direction in all inertial reference frames.

The first postulate points out that there is no "special" reference frame which is absolutely at rest and somehow 'better' than any other reference frame. It also implies that all the laws of physics (including Maxwell's equations) should take on the same form in all inertial reference frames. Again it says that there is no single inertial reference frame in which the laws of physics are simpler, or have fewer terms, than any other reference frame. The trick will be finding how to transform all of the coordinates from one inertial reference frame to another to preserve the form of the laws of physics. The second postulate codifies the results of the Michelson-Morley experiment, and leads to many non-intuitive results.

We examined the relativity of time by considering two reference frames, one with railroad tracks at rest (S), and the other (S') on a train moving down the tracks at a high rate of speed (V). Consider a light-clock on the train (frame S') that sends a brief flash of light from the floor to the ceiling, where it bounces off of a mirror, and then back to a detector that is co-located with the source on the floor. The time interval for the round trip of the light beam is $\Delta t' = 2h/c$, where h is the height of the train and c is the speed of light, as measured in S'. An observer (or really a set of observers) in S see the light follow a triangular trajectory as the train wizzes by. From the geometry of the experiment, and the second postulate, those observers attribute a time interval for the "round trip" of $\Delta t = \gamma \Delta t'$, where $\gamma = 1/\sqrt{1-\beta^2}$, and $\beta = V/c$. Since $\gamma > 1$ the two observers do not agree on how much time elapsed on the light-clock! This shows that the Galilean idea of universal time for all inertial observers is incorrect. In addition, because γ diverges as $V \to c$, it says that there is a speed limit for inertial reference frames: $V < c$. (This also means that we cannot address the question of what the world looks like from the reference frame of a photon travelling at the speed of light, at least with this formalism.)

The first postulate implies the equality of all inertial reference frames, so why is the result $\Delta t = \gamma \Delta t'$ asymmetric between the two inertial reference frames? The difference arises because the time interval was measured at a single fixed location in S' while it was measured at two distinct locations in S. The measurement of a time interval at a fixed location in an inertial reference frame is called the 'proper time interval' and is denoted Δt_0. Measurements of these two events taken from any other inertial reference frame moving with respect to this one will result in a dilated time interval measurement $\Delta t = \gamma \Delta t_0$.